Numerical Weather Prediction

Stability of the semi-implicit semi-lagrangian scheme proposed for the Unified Model

Forecasting Research Technical Report No. 182

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April 1996

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Abstract

This note describes investigations into the stability properties of the proposed semi-implicit semi-Lagrangian integration scheme for the Unified Model. The 2D, (x, y), shallow-water version of the model is stable at 96×65 resolution for all the timesteps tested, and produces acceptable results with very long timesteps, eg: 6 hours. The stability of the model in 2D, (x, z), simulations is then investigated and it is found that changes to the scheme as originally proposed are required to obtain stable integrations. Similar problems to those seen in the (x, z) model were also found in 3D simulations. The changes, which lead to a stable integration scheme in 3D as well as 2D, have been incorporated into version 9 of the proposed scheme as described in Cullen, Davies & Mawson (1996).

1 Introduction

There are many properties that can be required of a numerical method for the solution of an equation. In the case of numerical methods for solving the dynamical equations of the atmosphere our chief concerns are

- Spatial Accuracy
- Temporal Accuracy
- Stability
- Preservation of Global Integral Invariants

The first two items are self-explanatory, and the last is what is commonly referred to as *conservation*, though this is very loose and imprecise usage of the word. It is the third of the properties with which this note is concerned, and probably the most important for an operational forecast model. Stability is concerned with the growth of the solution modes of the equations. The following definition is used to describe a numerical method's stability properties.

Definition.:

Let s represent the numerical solution modes of a partial differential equation or system of partial differential equations which have analytic solution modes a such that

$$|a(t_1)| \leq |a(t)| \tag{1.1}$$

for time $t_1 > t$. Then given a timestep Δt calculate the amplification factor $\gamma \geq 0$ given by

$$|s(t + \Delta t)| = \gamma |s(t)| \tag{1.2}$$

for the numerical scheme. If

$\gamma < 1$ *then the scheme is said to be stable for timestep Δt, and if this is true for any choice of the timestep then the scheme is said to be unconditionally stable.*

$\gamma = 1$ *then the scheme is said to be neutrally stable.*

$\gamma > 1$ *then the scheme is said to be unstable for timestep Δt.*

If a there is a timestep beyond which the scheme is unstable but below which it is stable, then the scheme is said to be conditionally stable.

In general it is not possible to find an expression for γ for a numerical method applied to a system of partial differential equations, such as those used in atmospheric or ocean modelling. However it is often possible to find expressions for γ for parts of the system, or for the whole system by approximating certain of the terms by their linearised form. For non-hydrostatic atmospheric models there are three basic modes which need to be stable, the sound waves, the gravity waves, and the advective modes. For hydrostatic models, such as the current Unified Model, sound waves are not represented and hence only the other two modes remain. Proving stability of these modes for the linearised form does NOT prove that the scheme is stable, since non-linear effects could dominate. Currently the only way to investigate complete stability properties is by numerical experiment.

1.1 Current Unified Model stability properties

The current Unified Model is conditionally stable for both advective and gravity wave modes with the conditions being given by

$$\frac{u\Delta t}{\Delta x} < 1 \tag{1.3}$$

for the advective modes,

$$\frac{c\Delta t}{\Delta x} < 1 \tag{1.4}$$

for the gravity wave modes, where u is the maximum absolute wind speed, c is the external gravity wave speed, typically $330m/s$, Δx is the grid-length in meters and Δt the timestep in seconds. The values of the left-hand-side of equations (1.3) and (1.4) are known as the advective and gravity wave CFL numbers respectively. The grid-length Δx in the global model is given by $acos\phi\Delta\lambda$ where a is the radius of the earth, ϕ is the latitude in radians and $\Delta\lambda$ the east-west grid-spacing in radians. This expression changes with latitude and in the case of the gravity wave modes is largest on the row next to the pole. This condition enforces a severe restriction on the timestep that can be used for a given spatial resolution, a restriction that is alleviated in part in the Unified model by the use of time-splitting and Fourier damping (Cullen & Davies 1991). Time-splitting allows 3 timesteps of the terms involving the gravity wave to be performed to 1 for the advective terms. This makes the two conditions (1.3) and (1.4) almost identical since the maximum wind value in the atmosphere is typically $100m/s$. The Fourier damping decomposes the data on each row of the model into it's wave components and damps those waves which are unstable by multiplying them by a damping factor. It then recombines the resulting waves to obtain the damped data fields. This is done this way since the amplification factor γ has a dependence on the wave-number in each of the co-ordinate directions and for a given timestep and latitude the shorter wave lengths may be unstable whilst the longer wave-lengths are stable. Note that this does not remove the timestep limit it merely makes it less severe, since we could choose a timestep for data at the equator which makes all the waves unstable, and the Fourier damping would then reduce the data to a simple constant on each row. For the current climate model at 96×73 resolution such a timestep is of the order of 6

hours.

1.2 Expected properties of the semi-implicit semi-Lagrangian Unified model

It is expected that the semi-implicit semi-Lagrangian model will be unconditionally stable given the following provisos:

1. The time-weights α_i may need to be greater than 0.5, which is the stability limit given by the application of linear theory to the equations, because of non-linear instabilities (see Staniforth & Côté (1991) for a review of semi-Lagrangian advection and details about stability).

2. The variable coefficients which occur in the diagonal second-derivatives in the Helmholtz equation must be positive, ie: the coefficient which multiplies $\partial^2/\partial x^2$ must be positive and similarly for $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$.
   Note: This does not guarantee stability since it is a simplification of the full non-linear ellipticity condition and satisfaction of this simplified condition is not sufficient to satisfy the full condition. The use of the simplified condition could lead to a timestep limitation, however it is impractical to attempt to satisfy the full condition due to its complexity.

The following features of the scheme have not been tested elsewhere, to our knowledge, and require looking at.

1. The flux-form of the continuity equation has not been used in a semi-implicit semi-Lagrangian scheme before and this may cause a stability problem.

2. Currently existing semi-Lagrangian schemes advect temperature not potential temperature and this may also have implications for stability.

2 Shallow-water experiments

The full evaluation of the shallow-water version of the new integration scheme can be found in Malcolm (1996). Here we concentrate on two specific points, the formu-

lation of the continuity equation, and two different ways to discretize the advective terms in the momentum equations.

2.1 Continuity equation formulation

The continuity equations in flux form is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0 \tag{2.5}$$

where $\underline{u} = (u, v, w)$, ρ is the density, and we have written the equation in Cartesian form for simplicity. The flux form of the equation is inherently conservative in the sense that the global integral of the mass is conserved. This form of the continuity equation has not been used in previous semi-Lagrangian models, which have instead used the following, equivalent, form

$$\frac{D\rho}{Dt} + \rho \nabla.(\underline{u}) = 0 \tag{2.6}$$

where D/Dt represents the Lagrangian derivative which in Eulerian form is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}.\nabla$$

This form of the equation when time-discretised using the semi-Lagrangian approach is

$$\rho^{n+1} - \rho_d^n + [(1. - \alpha)\rho^n \nabla.(\underline{u}^n)]_d + \alpha \rho^n \nabla.\left(\underline{u}^{n+1}\right) = 0 \tag{2.7}$$

where subscript d denotes a value at the departure point of the trajectory, α is a time-weight, and ρ in the last term in the equation should be at time-level $n+1$ but is approximated by its time-level n value to avoid making the Helmholtz equation, which arises because of the semi-implicit scheme, non-linear. If we had left that ρ at time-level $n+1$ then a linear stability analysis implies that the scheme is stable if $\alpha \geq 0.5$. This form of the equation is non-conservative.

Equation 2.5 is discretised in time by

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + (1 - \alpha)\nabla.(\rho^n \underline{u}^n) + \alpha \nabla.\left(\rho^n \underline{u}^{n+1}\right) = 0 \tag{2.8}$$

where α is a time-weight and ρ in the last term in the equation should be at time-level $n+1$ but is approximated by its time-level n value to avoid making the Helmholtz

equation, which arises because of the semi-implicit scheme, non-linear. If we had left that ρ at time-level $n+1$ then a linear stability analysis implies that the scheme is stable if $\alpha \geq 0.5$.

Since in most atmospheric situations $\partial\rho/\partial t$ is very close to zero, then the approximation of ρ^{n+1} by ρ^n should not prove to be a restriction. This hypothesis is one area which required testing. In semi-Lagrangian models currently in use elsewhere the time-weighted terms are treated using a linearisation about a basic state and the non-linear term which arises from leaving ρ at time-level $n+1$ treated by extrapolation, see for example Ritchie, Temperton, Simmons, Hortal, Davies, Dent & Hamrud (1995). To test the flux-form of the continuity equation, the shallow-water version of the code was run for the NCAR test problems (Williamson, Drake, Hack, Jakob & Swarztrauber 1992), see Malcolm (1996) for the full results. Figure 1a shows the 500hpa height field after 5 days of integration using a 1800 second timestep at 96×65 resolution starting from data from 00Z 21st December 1978. The corresponding picture but using a 21600 second timestep is shown in figure 1b. Note that the only changes are a smoother solution with a loss of detail which is to be expected when the timestep is increased by a factor of 12. No spurious features such as extra depressions or anomalous highs are present and there is no sign of computational noise. These runs were performed with no diffusion and with the time-weight $\alpha = 0.5$ and the maximum value of the advective CFL is in excess of 20. This simulation would be impossible with the current Unified model at this resolution and with this timestep.

2.2 Momentum advection

Consider now just the advective part of the horizontal momentum equations, for simplicity we look just at u for the moment. The term to be considered is simply

$$\frac{Du}{Dt} = 0 \tag{2.9}$$

This can be discretised in semi-Lagrangian form by

$$u^{n+1} = u_d^n \tag{2.10}$$

where d is the departure point of the trajectory. For first order accuracy in time the departure point for the trajectory can be found by simply using $\underline{u}^n$. To obtain second order accuracy in time we can either use $\underline{u}^{n+1/2}$ or add an Eulerian correction to the semi-Lagrangian term calculated using $\underline{u}^n$ to find the departure point. The term with the Eulerian correction takes the form

$$u^{n+1} = u_d^n - 0.5\Delta t u' \frac{\partial u}{\partial x} - 0.5\Delta t v' \frac{\partial u}{\partial y} \tag{2.11}$$

where $u' = u^{n+1} - u^n$, and $v' = v^{n+1} - v^n$. This form would produce a highly coupled Helmholtz equation when used as part of a semi-implicit algorithm and is not computationally practical. The following form removes this complexity but still adds terms into the Helmholtz equation which increase the complexity of the coefficients found therein, note the different terms omitted in each wind component.

$$u^{n+1} = u_d^n - 0.5\Delta t v' \frac{\partial u}{\partial y} \tag{2.12}$$

$$v^{n+1} = v_d^n - 0.5\Delta t u' \frac{\partial v}{\partial x} \tag{2.13}$$

The alternative, to find the departure point using $\underline{u}^{n+1/2}$ is done by time-extrapolating the wind field to time-level $n+1/2$ using the following second-order accurate in time formula

$$\underline{u}^{n+1/2} = 1.5\underline{u}^n - 0.5\underline{u}^{n-1} \tag{2.14}$$

Figure 1b shows the solution with the time-extrapolated form, whilst figure 1c shows the solution with the Eulerian correction. Note that with the Eulerian correction spurious low pressure centres have developed in the southern hemisphere. Although the northern hemisphere solution is probably slightly better in shape it also has spuriously deep low pressure centres. The reason for this is the omission of the terms in equations (2.12)-(2.13) to avoid the coupling in the Helmholtz equation. The omission of these terms produces some spurious vorticity sources and hence the spuriously deep depressions. Simulations at shorter timesteps, eg: 1800 seconds, do not show this behaviour to be as drastic and verify slightly better, but probably because the spurious vorticity sources, though weaker, are helping to maintain the central pressures of the lows. The time-extrapolated version has been adopted for version 9 of the New Integration Scheme. A further advantage of this approach is

that it simplifies some of the coefficients in the Helmholtz equation, making it more strongly elliptic.

3 2D (x, z) Experiments

Experiments with the 3D model as described in Cullen, Davies & Mawson (1995) at $96 \times 61 \times 20$ resolution showed instability when the horizontal advective CFL number exceeded 1. To help in understanding this limitation the 2D (x, z) model was run to see if it suffered from a similar restriction. The test problem chosen was the Eady wave simulation, see Cullen & James (1994) for details of the test problem. Figure 2 shows the wind and potential temperature fields after 3 days of integration with a 7200 second timestep and with the time-weight α set to the maximum value of 1. It is immediately obvious that there is an instability in the model, the oscillations in the vertical velocity field stand out but there is also evidence of waves in both the potential temperature (theta) and u fields. These oscillations suggested problems with the vertical part of the scheme, particularly the buoyancy terms since it was known that the horizontal part of the scheme was stable from the shallow-water tests. The terms involved are the vertical advection of potential temperature and the vertical pressure gradient term.

In version 8 of the new integration scheme (Cullen et al. 1995) three different definitions of the potential temperature in time level $n + 1$ are used, namely

$$\theta^{n+1} = \theta_d \tag{3.15}$$

$$= \theta^n - \Delta t(\underline{u}^n + \underline{u}')\nabla\theta \tag{3.16}$$

$$= \theta_{dn} - \Delta t \alpha w' \frac{\partial \theta^n}{\partial z} \tag{3.17}$$

The final advection of the potential temperature, θ, was given by equation (3.15) where the trajectory was given by $\underline{u}^n + \underline{u}'$. To obtain the $\underline{u}'$ terms from the Helmholtz equation an expression for θ^{n+1} is required in the equation of state. This was given by equation (3.16) rather than the semi-Lagrangian form used for the update. The $\underline{u}'\nabla\theta$ is used to make the Helmholtz equation solvable, since it is currently not practical to include trajectory calculations inside elliptic equation solvers. The $\underline{u}\nabla\theta$

term is used to obtain enegetic consistency. The vertical pressure gradient term was given by

$$\left[(1-\alpha)c_p\theta^n\frac{\partial\Pi^n}{\partial z}\right]_{dn}+\alpha c_p\theta^{n+1}\frac{\partial\Pi^{n+1}}{\partial z} \tag{3.18}$$

where $\Pi=(p/p_0)^\kappa$ is the Exner pressure and θ^{n+1} is given by equation (3.17) where the departure point dn was calculated from the trajectory given by $\underline{u}^n$. This definition is used to avoid coupling together the correction equations for u' and v' with that for w' resulting in a simpler Helmholtz equation. Note that there are three different definitions of θ^{n+1} being used and this could be leading to an instability due to the differences in the three resulting values of θ^{n+1}. The three definitions were being used to obtain energetic consistency, and also to avoid complications in the Helmholtz equation which would have made it too difficult and expensive to solve numerically given current numerical techniques.

Experiment 1.

As a first attempt to remove the instability the three definitions were reduced to one, namely the simplest one given by equation (3.17). This had no significant impact on the instability.

Experiment 2.

A further experiment was tried by once again using just one definition for θ^{n+1} but this time using

$$\theta^{n+1}=\theta_{dl}-\Delta t(1-\alpha)\left[(w^n-w^*)\frac{\partial\theta^n}{\partial z}\right]_{dl}-\Delta t\alpha(w^n+w'-w^*)\frac{\partial\theta^n}{\partial z} \tag{3.19}$$

where the departure point dl is on the nearest model level to the departure point given by trajectory $\underline{u}^n$, and

$$w^*=(z_a-z_{dl})/\Delta t$$

is the velocity required to move the parcel from the arrival point a to the departure point dl in one timestep. This form of advection is known as *non-interpolating in the vertical semi-Lagrangian advection.* The results for day 3 of the integration are

shown in figure 3. Note the complete absence of instability in these pictures, this is despite the fact that the time-weight α is set 0.5, the minimum possible value for linear stability. The figures for day 5 of the same integration are shown in figure 4, and still show no sign of instability. Using this formulation in the 3D version of the model also removed the instability problems previously seen therein.

Question: Why does the formulation of experiment 2 remove the instability present in both experiment 1 and the original formulation of the scheme ?

The primary difference between the three experiments is in the treatment of the advection of the time-level n potential temperature, and not in the treatment of the advection of the correction term $\underline{u}'\nabla\theta$. It should be noted that the instability appears when the horizontal CFL number exceeds 1 but with the vertical CFL number $w\Delta t/\Delta z$ significantly less than 1. Differentiating the equation of state,

$$p = R\rho\Pi\theta \tag{3.20}$$

where R is the gas constant and $\Pi = (p/p_0)^\kappa$ is the Exner pressure, with respect to z and using the relationship

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = c_p\theta\frac{\partial \Pi}{\partial z} \tag{3.21}$$

gives, after some manipulation

$$c_p\theta\frac{\partial \Pi}{\partial z} = \frac{Rp}{\rho^2(1-\kappa)}\frac{\partial \rho}{\partial z} - \frac{R\Pi}{(1-\kappa)}\frac{\partial \theta}{\partial z} \tag{3.22}$$

Substituting this into equation (3.18) gives the following equivalent form for the vertical pressure gradient equation

$$(1-\alpha)\left[\frac{Rp}{\rho^2(1-\kappa)}\frac{\partial \rho}{\partial z} - \frac{R\Pi}{(1-\kappa)}\frac{\partial \theta}{\partial z}\right]_{dn} + \alpha\left[\frac{Rp^{n+1}}{\rho^{n+1^2}(1-\kappa)}\frac{\partial \rho^{n+1}}{\partial z} - \frac{R\Pi^{n+1}}{(1-\kappa)}\frac{\partial \theta^{n+1}}{\partial z}\right] \tag{3.23}$$

It would seem reasonable to expect that using the numerical approximation of $\partial\theta/\partial z$ in the vertical pressure gradient term, equation (3.23), and in the advection of potential temperature would produce a more consistent scheme, but it is not obvious that using different approximations should cause instability. The numerical approximations used in the non-interpolating in the vertical semi-Lagrangian advection of

potential temperature, equation (3.19), are the same as used in the vertical pressure gradient term. If we assume that the problem is the representation of the vertical derivative of the potential temperature then it is easy to see why increasing horizontal CFL number can increase the difference between the interpolating semi-Lagrangian representation and the representation in the pressure gradient term, and why the differences between the non-interpolating semi-Lagrangian representation and the representation in the pressure gradient term are small. The representation of $\partial\theta/\partial z$ in the pressure gradient term is an average of the values at the arrival point and at the departure point. The representation of $\partial\theta/\partial z$ in the interpolating scheme is always just at the departure point. Thus as the departure point gets further away from the arrival point, the differences between the representations gets potentially larger. In the non-interpolating semi-Lagrangian scheme, the representation of $\partial\theta/\partial z$ is the same average of the values at the departure and arrival points, and so is not affected by the departure point being further away in the horizontal. This instability has not been found at other centres using semi-Lagrangian advection, for example ECMWF (Ritchie et al. 1995), since they advect temperature rather than potential temperature. When advecting temperature the semi-Lagrangian equation is

$$T^{n+1} - T^n = \left[(1-\alpha)\Delta t \frac{\kappa T^n \omega^n}{p^n}\right]_d + \left[\alpha \Delta t \frac{\kappa T^{n+1}\omega^{n+1}}{p^{n+1}}\right] \qquad (3.24)$$

The right-hand-side of equation (3.24) is implicit in the vertical advection of potential temperature. Note that this term is again treated as an average of the values at the arrival and departure points. The $\kappa T\omega/p$ term in the advection of T is contained within the $w\partial\theta/\partial z$ term in the advection of θ.

Another possible reason is that the use of $w\partial\theta/\partial z$ is more consistent with the update of p_*, the surface pressure, by $\nabla.(\rho\underline{u}$ which would be applicable in a hydrostatic model. The $w\partial\theta/\partial z$ is the mass term for internal gravity wave modes and inconsistencies could lead to gravity wave instabilities. Note that in hydrostatic semi-Lagrangian models used elsewhere p_* is advected by the semi-Lagrangian scheme but the divergence term $p_*\nabla.\underline{u}$ is treated as either an average along the trajectory or extrapolated in time to the mid-point of the trajectory.

4 Conclusions and Recommendations

Conclusions:

- The use of the flux form of the continuity equation has no detrimental affect on model stability. Moreover it guarantees formal conservation of mass which has not been possible in previous semi-Lagrangian models.

- The use of time-extrapolated trajectories to calculate the departure point is preferable to the use of Eulerian advective corrections for long timesteps. For shorter timesteps the differences are minimal.

- The interpolating semi-Lagrangian scheme for potential temperature is unstable for horizontal CFL numbers of around 1 and larger. This instability is related to the treatment of the vertical advection of potential temperature. The instability may be due to inconsistencies in treating the $\partial\theta/\partial z$ term in the advection of potential temperature and its equivalent term in the vertical pressure gradient term. It may also relate to the treatment of the term which would appear on the right-hand-side of the equation for the advection of temperature which is implicit in the vertical advection of potential temperature. The non-interpolating in the vertical semi-Lagrangian advection of potential temperature removes this problem.

Recommendations:

- The use of time-extrapolated trajectories to calculate the departure point is adopted.

- The non-interpolating in the vertical semi-Lagrangian advection of potential temperature is adopted.

- Only one definition of the potential temperature at time-level $n+1$ is assumed in forming the Helmholtz equation and that it is form given by equation (3.19). NB: This could cause a problem where $\partial\theta/\partial z$ is small and the horizontal gradients are quite large. In this case the model would be unable to easily adjust the potential temperature to satisfy the equation of state using equation (3.19),

but would be able to if the horizontal corrections $u'\partial\theta/\partial x$ and $v'\partial\theta/\partial y$ were included. The problem with including these terms is that they would lead to a much more complicating and expensive Helmholtz equation if only one definition of the potential temperature at time-level $n+1$ is allowed in forming the Helmholtz equation. The compromise position would be once again allow multiple definitions of the potential temperature at time-level $n+1$ in forming the Helmholtz equation, but this could lead to stability problems. This situation will be reviewed if it is found that the recommended choice is inappropriate.

This recommendations have been incorporated into version 9 of the new integration scheme (Cullen et al. 1996).

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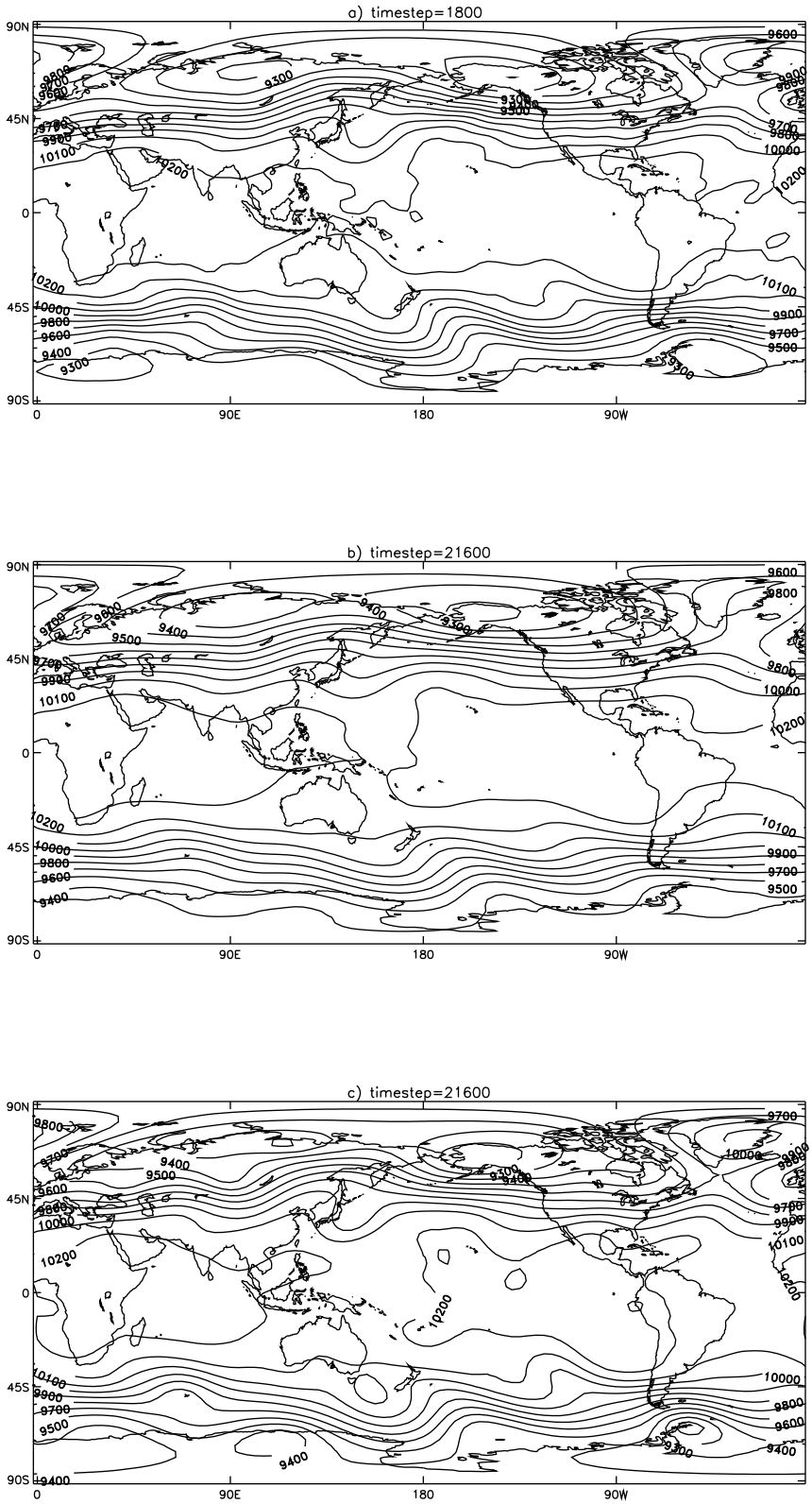

Figure 1: Comparison of shallow-water formulations.

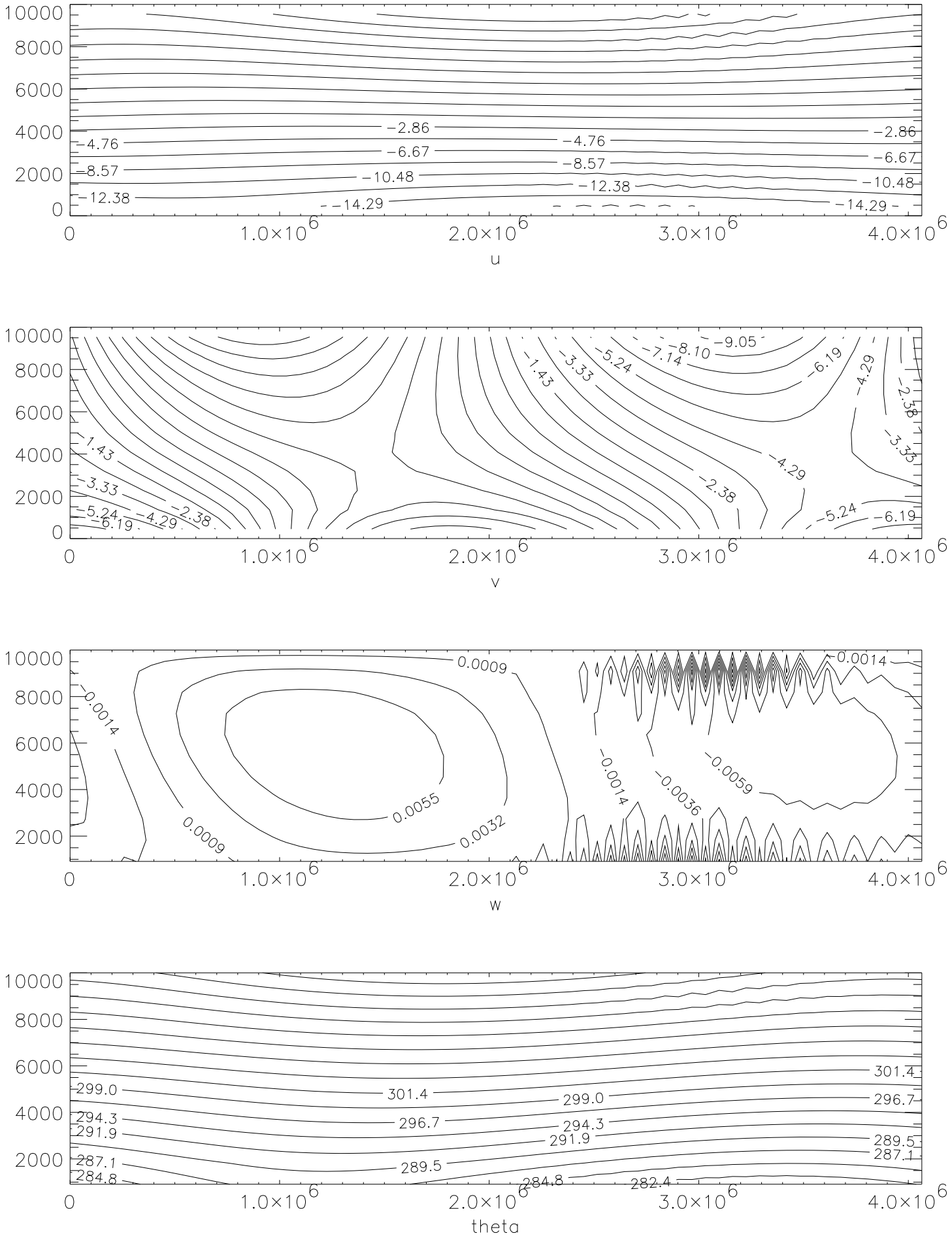

Figure 2: Version 8 formulation; day 3 figures.

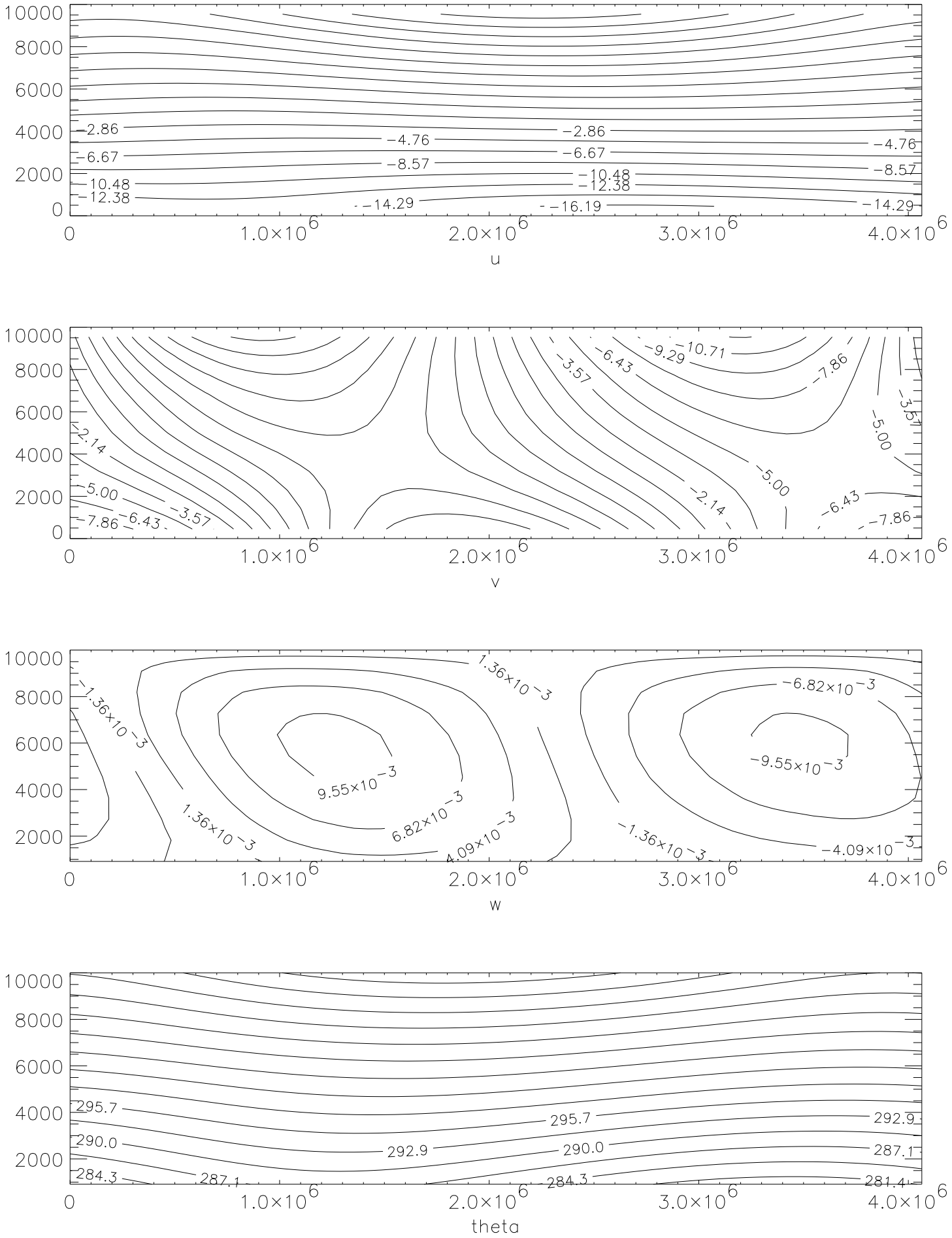

Figure 3: Experiment 2 formulation; day 3 figures.

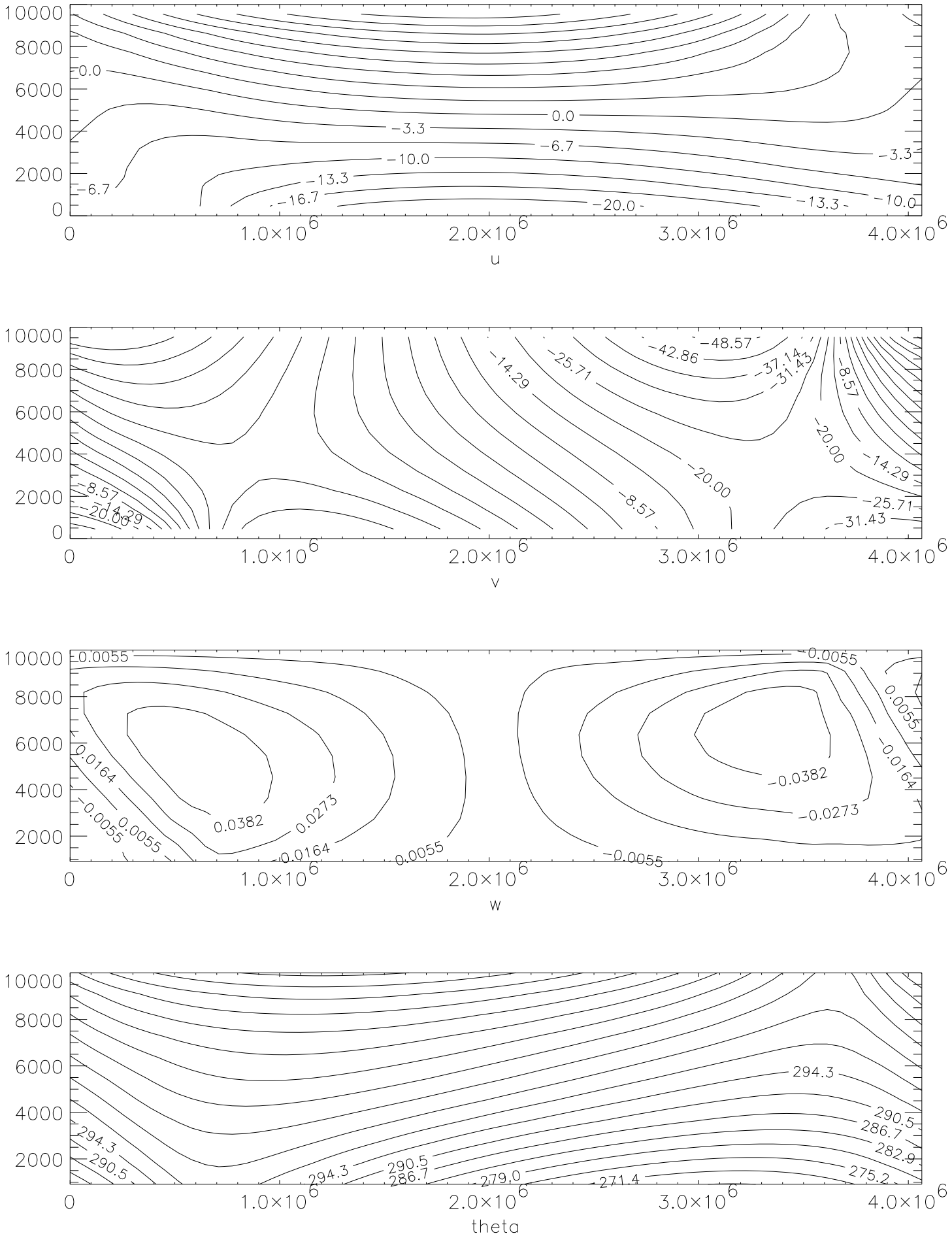

Figure 4: Experiment 2 formulation; day 5 figures.